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# Analyticity of Event Horizons of Five-Dimensional Multi-Black Holes with Non-Trivial Asymptotic Structure

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## Abstract

We show that there exist five-dimensional multi-black hole solutions which have analytic event horizons when the space-time has non-trivial asymptotic structure, unlike the case of five-dimensional multi-black hole solutions in asymptotically flat space-time.

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*Introduction.*— Analyticity of the metric is crucial property if one wants to investigate global structure of the space-time extended across the horizons. For example, it is well known that Schwarzschild solution has unique analytic extension across the event horizon. There are many exact solutions which have analytic horizons [1] [1], and the natural generalizations of four-dimensional single black holes in asymptotic flat space-time to higher dimensions [2, 3] have also analytic horizons.

In the case of multi-black holes, four-dimensional Majumdar-Papapetrou(MP) solutions [6, 7] have analytic event horizons [8]. In contrast, the event horizons of five or higher-dimensional MP solutions [9] are not analytic. This was first investigated by Gibbons et al [10] and Welch [11], and recently reinvestigated by Candlish and Reall [12]. These works would suggest event horizons of multi-black holes in higher dimensions are non-analytic in general. However this is not true if the space-time has non-trivial asymptotic structure as discussed later.

In this paper, we focus on five-dimensional space-times whose topology of boundary of spatial infinity is not $\mathrm{S}^3$ but the lens space, $\mathrm{S}^3/\mathbb{Z}_n$, and we give simple examples of five-dimensional multi-black hole solutions which have analytic event horizons.

*Four-Dimensional Multi-Black Holes in Asymptotically Flat Space-Time.*— First we briefly review that event horizons of the multi-black holes in four-dimensional asymptotically flat space-time, i.e., four-dimensional MP solutions are analytic. For simplicity, we restrict ourselves to the MP solutions with two black holes given by

$$ds^2 = -H^{-2}dt^2 + H^2(dr^2 + r^2 d\Omega^2_{\mathrm{S}^2}), \tag{1}$$

$$H = 1 + \frac{m_1}{r} + \frac{m_2}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}. \tag{2}$$

To remove the divergence of $g_{rr}$ at the event horizon $r = 0$, we introduce a coordinate across the horizon as

$$dt = du + H^2 dr + W(r,\theta)d\theta, \tag{3}$$

$$W(r,\theta) = \int \frac{\partial(H^2)}{\partial\theta} dr. \tag{4}$$

Then the metric (1) becomes

$$ds^2 = -H^{-2}du^2 - 2dudr - 2H^{-2}W dud\theta - 2W drd\theta + W^2 d\theta^2 + H^2 r^2 d\Omega^2_{\mathrm{S}^2}. \tag{5}$$

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[1] In more complicated situation, some exact solutions have non-analytic horizons [4, 5].

In the neighborhood of the horizon $r=0$, the functions $H$, $H^{-1}$ and $W$ behaves as

$$H = \frac{m_1}{r} + O(r^0), \tag{6}$$

$$H^{-1} = \frac{r}{m_1} + O(r^2), \tag{7}$$

$$W = -\frac{2m_1 m_2 \sin\theta}{a^2} r + O(r^2), \tag{8}$$

where we choose integral constant of $W$ given by (4) as zero. Then the metric at the horizon becomes

$$ds^2 = -2dudr + m_1^2 d\Omega^2_{\mathrm{S}^2}. \tag{9}$$

So we can see this coordinate covers the horizon $r=0$. Moreover, one can easily check all the metric components in this coordinate are analytic function of $r$ at the horizon $r=0$, therefore the extension by use of (3) and (4) is analytic extension across the horizon $r=0$.

*Five-dimensional Multi-black holes in asymptotically flat space-time.*— Next, in the case of five-dimensional MP solutions, we see that the coordinate across the horizon which is introduced like the case of four-dimensional MP solutions fails to be analytic at the horizon. The metric of five-dimensional MP solution with two black holes is given by

$$ds^2 = -H^{-2}dt^2 + H(dr^2 + r^2 d\Omega^2_{\mathrm{S}^3}), \tag{10}$$

$$H = 1 + \frac{m_1}{r^2} + \frac{m_2}{r^2 + a^2 - 2ar\cos\theta}. \tag{11}$$

Similar to the equation (3), we introduce a coordinate as

$$dt = du + H^{3/2}dr + W(r,\theta)d\theta, \tag{12}$$

$$W(r,\theta) = \int \frac{\partial (H^{3/2})}{\partial\theta} dr. \tag{13}$$

Then the metric becomes

$$\begin{aligned} ds^2 = & -H^{-2}du^2 - 2H^{-1/2}dudr \\ & -2H^{-2}Wdud\theta - 2H^{-1/2}Wdrd\theta - H^{-2}W^2d\theta^2 + Hr^2 d\Omega^2_{\mathrm{S}^3}. \end{aligned} \tag{14}$$

In the neighborhood of the horizon $r=0$, the functions $H$, $H^{-1}$ and $W$ behaves as

$$H = \frac{m_1}{r^2} + O(r^0), \tag{15}$$

$$H^{-1} = \frac{r^2}{m_1} + O(r^4), \tag{16}$$

$$W = -\frac{3\sqrt{m_1} m_2 \sin\theta}{a^3} r + O(r^2), \tag{17}$$

then all the metric components do not diverge at the horizon. However, unlike the case of four-dimensional MP solution, the metric component $g_{ur}$ is zero at $r = 0$. The horizon $r = 0$ is still coordinate singularity since the metric is degenerate at the horizon in this coordinate. To remove this coordinate singularity we further introduce the new radial coordinate as

$$\tilde{r} := r^2, \tag{18}$$

then the metric becomes

$$\begin{aligned} ds^2 \;=\; & -H^{-2}du^2 - H^{-1/2}\tilde{r}^{-1/2}dud\tilde{r} \\ & -2H^{-2}Wdud\theta - H^{-1/2}\tilde{r}^{-1/2}Wd\tilde{r}d\theta - H^{-2}W^2d\theta^2 + H\tilde{r}d\Omega^2_{\mathrm{S}^3}. \end{aligned} \tag{19}$$

The behaver of the metric at the horizon is

$$ds^2 \;=\; -m_1^{-1/2}dud\tilde{r} + m_1 d\Omega^2_{\mathrm{S}^3}, \tag{20}$$

then we can see this coordinate covers the horizon $\tilde{r} = 0$. However, in this coordinate the function $H$ becomes

$$H = 1 + \frac{m_1}{\tilde{r}} + \frac{m_2}{\tilde{r} + a^2 - 2a\sqrt{\tilde{r}}\cos\theta}, \tag{21}$$

and we can see the third term in the right hand side of (21) is not analytic function of $\tilde{r}$ at the horizon $\tilde{r} = 0$ because of the existence of $\sqrt{\tilde{r}}$ in the denominator. This is the essential reason why analyticity is broken in five-dimensional case unlike the four-dimensional case. In fact, from more careful discussions [11, 12], it is shown that there is no coordinate in which the metric function is analytic at the event horizon.

*Five-Dimensional Multi-black holes with non-trivial asymptotic structure.*— Finally, we investigate analyticity of horizons of multi-black holes constructed on the Gibbons-Hawking space which has non-trivial asymptotic structure, i.e., the topology of the spatial infinity is lens space $\mathrm{S}^3/\mathbb{Z}_n$ [13, 14, 15]. The metric with two black holes is given by

$$ds^2 \;=\; -H^{-2}dt^2 + H ds^2_{\mathrm{GH}}, \tag{22}$$

$$ds^2_{\mathrm{GH}} \;=\; V^{-1}\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) + V\left(d\zeta + \omega_\phi d\phi\right)^2, \tag{23}$$

$$H \;=\; 1 + \frac{M_1}{r} + \frac{M_2}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}, \tag{24}$$

$$V^{-1} \;=\; \epsilon + \frac{N_1}{r} + \frac{N_2}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}, \tag{25}$$

$$\omega_\phi \;=\; N_1\cos\theta + \frac{N_2(-a + r\cos\theta)}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}, \tag{26}$$

where $ds^2_{\rm GH}$ denotes the metric of the Gibbons-Hawking space [16], which reduces to the Eguchi-Hanson space for the case $\epsilon = 0$ and to the multi-Taub-NUT space for the case $\epsilon = 1$ [2]. Similar to the cases discussed above, to remove the divergence of $g_{rr}$ at the horizon $r = 0$, we introduce a coordinate as

$$
\begin{aligned}
du &= dt + H^{3/2}V^{-1/2}dr + W(r,\theta)d\theta, && (27)\\
W(r,\theta) &= \int \frac{\partial(H^{3/2}V^{-1/2})}{\partial\theta}dr. && (28)
\end{aligned}
$$

Then the metric becomes

$$
\begin{aligned}
ds^2 &= -H^{-2}du^2 - 2H^{-1/2}V^{-1/2}dudr\\
&\quad -2H^{-2}Wdud\theta - 2H^{-1/2}V^{-1/2}Wdrd\theta - H^{-2}W^2d\theta^2\\
&\quad +r^2HV^{-1}d\theta^2 + r^2\sin^2\theta HV^{-1}d\phi^2 + HV\left(d\zeta + \omega_\phi d\phi\right)^2. && (29)
\end{aligned}
$$

In the neighborhood of the horizon $r = 0$, the functions $H$, $H^{-1}$, $V^{-1}$, $V$, $W$ and $\omega_\phi$ behave as

$$
\begin{aligned}
H &= \frac{M_1}{r} + O(r^0), && (30)\\
H^{-1} &= \frac{r}{M_1} + O(r^2), && (31)\\
V^{-1} &= \frac{N_1}{r} + O(r^0), && (32)\\
V &= \frac{r}{N_1} + O(r^2), && (33)\\
W &= -\sqrt{\frac{M_1}{N_1}}\frac{(3M_2N_1 + M_1N_2)\sin\theta}{2a^2}r + O(r^2), && (34)\\
\omega_\phi &= N_1\cos\theta - N_2 + O(r), && (35)
\end{aligned}
$$

then all the metric components do not diverge at the horizon. Unlike the five-dimensional MP solution, the metric component $g_{ur}$ is non-zero at the horizon because of the existence of $V^{-1}$. Actually, the metric behaves at the horizon as

$$
ds^2 = -2\sqrt{\frac{N_1}{M_1}}dudr + M_1N_1\left[d\theta^2 + \sin^2\theta d\phi^2 + \left(\frac{d\tilde{\zeta}}{N_1} + \cos\theta d\phi\right)^2\right], \tag{36}
$$

where $\tilde{\zeta} = \zeta - N_2\phi$, then this coordinate covers the horizon $r = 0$ [3]. Moreover, the metric is analytic function of $r$ at the event horizon $r = 0$ like the four-dimensional MP solution.

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[2] In the special case $\epsilon = 1$ and $H = V^{-1}$, then the metric (22) reduces to four-dimensional MP solutions with a twisted $S^1$ bundle. In this case, the horizons are clearly analytic because four-dimensional MP solutions are analytic on the horizon.

[3] The spatial cross section of the event horizon in equation (36) is $\mathrm{S}^3$ or the lens space $\mathrm{S}^3/\mathbb{Z}_n$ [14, 15].

Therefore this extension is an analytic extension across the horizon $r = 0$. From this, we can see that there exist five-dimensional multi-black hole solutions which have analytic event horizons when the space-time has non-trivial asymptotic structure. This fact suggests that the analyticity of the event horizon is tightly related to the asymptotic structure of the space-time.

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